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N-Fold Backlund Transformation for Deformed Nonlinear Schrödinger Equation

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We formulate the N-fold Backlund transformation for the deformed nonlinear Schrödinger equation following the idea of Neugebauer *et al.* Starting with trivial solutions, we construct explicit one-soliton solutions in two distinct cases of deformation from our transformation for $N = 1$. These solitons have space-time-dependent amplitude and velocity.

1. INTRODUCTION

The Backlund transformation (BT) is a useful tool for the generation of multisoliton states of nonlinear integrable systems (Miupa, 1976). Recently various methods have been suggested for the construction of Backlund transformation (Matveev and Salle, 1991). The BT usually works sequentially for the generation of the one-soliton state from the trivial one, the two-soliton solution from the one-soliton solution, and so on. But Neugebauer and Meinel (1984) proposed a method by which it became possible to construct an N-fold BT which can produce a general N-soliton solution by starting from a trivial solution in a single stroke. Such a methodology has been applied in the case of the self-dual Yang–Mills equation and chiral field equations (Pohle, 1984) and in some problems of general relativity with commuting Killing vectors.

On the other hand, recently Zakharov *et al.* (1987) showed that if we consider the eigenvalue λ of the spectral problem to be a function of space and time, then new (x, t)-dependent integrable systems can be generated. [For earlier work see Gupta *et al.* (1979).] But as the eigenvalue λ becomes a function of (x, t), it becomes pointless to speak of analyticity in λ, which

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is actually the basis of the inverse problem. In a recent work De and Roy Chowdhury (1991) show that by invoking the analyticity in the "constant part" of the parameter λ (which is now a function of x, t) it is possible to reconstruct the nonlinear field through a Riemann–Hilbert problem.

In this paper we apply the formalism of Neugebauer to construct an N-fold Backlund transformation for the deformed nonlinear Schrödinger (NLS) equation and show that for $N = 1$ it gives correctly the one-soliton solution whose amplitude and speed may be functions of x and t. Among the two types of deformations of the NLS equation, the first is simple in the sense that by a proper change of dependent and independent variable we can map this equation to the original NLS equation and the final result of the soliton solution can be checked directly. On the other hand, the second type of deformation leads to a new (x, t)-dependent NLS equation for which it is not possible to construct such a transformation.

2. FORMULATION

The two types of nonlinear Schrödinger equation under consideration are written as follows:

Case (a):

$$i\left(q_t + \frac{q}{2t}\right) + q_{xx} \pm 2|q|^2 q = 0 \tag{1}$$

This is sometimes referred to as the cylindrical NLS equation.

Case (b):

$$iq_t + q_{xx} + \frac{q_x}{x} \pm 2|q|^2 q = \frac{q}{x^2} \mp 4q \int \frac{|q|^2}{x}\, dx \tag{2}$$

The Lax pair associated with these equations is written as

$$\Phi_x = u\Phi, \qquad \Phi_t = V\Phi \tag{3}$$

with

$$u = i\lambda\sigma_3 + iq\sigma_+ \pm i\bar{q}\sigma_- \tag{4}$$

for both cases (a) and (b), and the corresponding time parts

$$V^{(a)} = -2i\lambda^2\sigma_3 - 2i\lambda(q\sigma_+ \pm \bar{q}\sigma_-) + (\pm i|q|^2\sigma_3 - q_x\sigma_+ \pm \bar{q}_x\sigma_-) \tag{5}$$

whence $\lambda_x = 1/4t$ and $\lambda_t = -\lambda/t$, so that

$$\lambda = (Z + x/4)(1/t) \tag{6}$$

and

$$V^{(b)} = -2i\lambda^2\sigma_3 - 2i\lambda(q\sigma_+ \pm q\sigma_-) + \left[\left(i|q|^2 + 2i\int \frac{|q|^2}{x}\,dx\right)\sigma_3 - \left(q_x + \frac{q}{x}\right)\sigma_+ \pm \left(\bar{q}_x + \frac{\bar{q}}{x}\right)\sigma_-\right] \tag{7}$$

along with

$$\lambda_x = \frac{\lambda}{x}\,\lambda_t = -\frac{4\lambda^2}{x}$$

so that

$$\lambda = \frac{x}{4(z + t)} \tag{8}$$

In each case the spectral parameter λ becomes a function of (x, t), as shown in equations (6) and (8). Here Z is a constant of integration. In M. De (1991) we have shown that it is possible to prove the analyticity properties of the Jost functions in the variable Z.

Let Φ^0 be an initially known eigenfunction corresponding to a trivial solution q_0 of either (1) or (2) and let the corresponding Lax matrices be denoted as u^0 and V^0. We set out to construct a new solution Φ such that

$$\Phi = P\Phi^0 \tag{9}$$

whence Φ will obey again equation (3). Here P is a 2×2 matrix function which satisfies the following conditions.

(i) P is a polynomial in Z with matrix coefficients Q:

$$P = P(Z) = \begin{bmatrix} P_{11}(Z) & P_{12}(Z) \\ P_{21}(Z) & P_{22}(Z) \end{bmatrix} = \sum_{j=0}^{N} Q_j(xt)Z^j \tag{10}$$

(ii) $P(\bar{Z})M(\bar{Z}) = \overline{M(Z)P(Z)}$, where

$$M(Z) = \begin{pmatrix} 0 & \mu + Z \\ \mu - Z & 0 \end{pmatrix} \quad \text{for real constant } \mu \tag{11}$$

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for imaginary constant } \mu$$

(iii) We have

$$(0, 1) \quad P(Z = \mu)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \tag{12}$$

(iv) We have

$$\text{Det } Q_N = 1 \tag{13}$$

(v) $Z = Z_j$, for $j = 1, \ldots, N$, are the zeros of Det $P(Z)$ provided that Det $\Phi^0(Z)$ are regular and there exists a scalar potential factor b_j, $j = 1, \ldots, N$, such that $\hat{\phi}(Z_j) = b_j \hat{\varphi}(Z_j)$ at the zeros of Det $P(Z)$. Here

$$\Phi = [\hat{\phi}(Z); \hat{\varphi}(Z)] = \begin{bmatrix} \phi_1(xtz) & \varphi_1(xtz) \\ \phi_2(xtz) & \varphi_2(xtz) \end{bmatrix} \tag{14}$$

(vi) There is a gauge freedom for P so that P transforms to P' via

$$P' = SP \qquad \text{with} \quad S = \begin{pmatrix} e^{i\chi(xt)} & 0 \\ 0 & e^{-i\chi(xt)} \end{pmatrix}$$

Now from conditions (ii) and (iii) we immediately deduce

$$P_{12}(\bar{Z}) = \frac{M_{12}(\bar{Z})}{M_{21}(\bar{Z})} \overline{P_{21}(Z)} \tag{15}$$

and $P_{11}(\bar{Z}) = \overline{P_{22}(Z)}$. Also from the condition that the P_{ij} are polynomial in Z we obtain

$$\begin{aligned} P_{12}(Z) &= (\mu + Z)(\bar{a}_0 + \bar{a}_1 Z + \bar{a}_2 Z^2 + \cdots + \bar{a}_{N-1} Z^{N-1}) \\ P_{21}(Z) &= (\mu - Z)(a_0 + a_1 Z + a_2 Z^2 + \cdots + a_{N-1} Z^{N-1}) \\ P_{11}(Z) &= \bar{A}_0 + \bar{A}_1 Z + \bar{A}_2 Z^2 + \cdots + \bar{A}_N Z^N \\ P_{22}(Z) &= A_0 + A_1 Z + A_2 Z^2 + \cdots + A_N Z^N \end{aligned} \tag{16}$$

It may now be observed that altogether there are $2N + 1$ unknowns (functions of x and t). In the ensuing discussions we will set up equations for determining these functions.

Let us define α_j by

$$\alpha_j = \frac{\phi_1^0(Z_j) - b_j \varphi_1^0(Z_j)}{\phi_2^0(Z_j) - b_j \varphi_2^0(Z_j)} \tag{17}$$

where Z_j ($j = 1, 2, \ldots, N$) are zero of Det $P(Z)$ and b_j = const. Then using

$$\Phi_x^0(Z_j) = u^0(Z_j)\Phi^0(Z_j) \tag{18}$$

we get for case (a)

$$\alpha_{jx} = 2i\left(\frac{Z_j}{t} + \frac{x}{4t}\right)\alpha_j + iq_0 \mp i\bar{q}_0 \alpha_j^2 \tag{19}$$

where q_0 is the solution of the nonlinear system corresponding to Φ^0. Similarly

$$\alpha_{jt} = \left[-4i\left(\frac{Z_j}{t} + \frac{x}{4t}\right)^2 \pm 2i|q_0|^2\right]\alpha_j + \left[\pm 2i\bar{q}_0\left(\frac{Z_j}{t} + \frac{x}{4t}\right)\right.$$
$$\left.\mp \bar{q}_{0x}\right]\alpha_j^2 - 2iq_0\left(\frac{Z_j}{t} + \frac{x}{4t}\right) - q_{0x} \tag{20}$$

For case (b) we get

$$\alpha_{jx} = iq_0 + \frac{ix}{2(Z_j + t)}\alpha_j \mp i\bar{q}_0\alpha_j^2$$
$$\alpha_{jt} = \left(\pm\frac{2ix}{4(Z_j + t)}\bar{q}_0 \mp \bar{q}_{0x} \mp \frac{\bar{q}_0}{x}\right)\alpha_j^2 \tag{21}$$
$$+ \left(-\frac{4ix^2}{16(Z_j + t)^2} \pm 2i|q_0|^2 \pm 4i\int\frac{|q_0|^2}{x}\,dx\right)\alpha_j$$
$$+ \left(-\frac{2ix}{4(Z_j + t)}q_0 - q_{0x} - \frac{q_0}{x}\right)$$

Now let us go back to the condition

$$\Phi(xtZ) = P(Z)\Phi^0(xtZ) \tag{22}$$

We now impose the condition that the zeros of Det $P(Z)$, that is, the Z_j, all are independent of (x, t). Now from equation (22) we have

$$\text{Det}\,\Phi = \text{Det}\,P\ \text{Det}\,\Phi^0 \tag{23}$$

whence Det $\Phi(Z_j) = 0$, implying

$$\phi_1(Z_j) = b_j\varphi_1(Z_j) \qquad \text{and} \qquad \phi_2(Z_j) = b_j\varphi_2(Z_j)$$

along with the condition

$$P_{21}(Z_j)[\phi_1^0(Z_j) - b_j\varphi_1^0(Z_j)] + P_{22}(Z_j)[\phi_2^0(Z_j) - b_j\varphi_2^0(Z_j)] = 0$$

or

$$P_{21}(Z_j)\alpha_j + P_{22}(Z_j) = 0 \tag{24}$$

where α_j is the quantity defined in equation (17). With the explicit forms of P_{ij} given in equation (16), equation (24) leads to conditions for the determination of the coefficients a_j, A_j, etc.

3. DETERMINATION OF P_{ij}

It is evident from the previous analysis that the most important aspect of our formulation is the construction of the coefficients of the matrix P_{ij}.

Substituting from equation (16) in (24), we get, after dividing by a_{N-1},

$$X_0 + X_1 Z_j + X_2 Z_j^2 + \cdots + N_X Z_j^N + Y_0 \beta_j + Y_1 \beta_j Z_j + Y_2 \beta_j Z_j^2 + Y_{N-2} \beta_j Z_j^{N-2} + \beta_j Z_j^{N-1} = 0 \tag{25}$$

for $j = 1, 2, \ldots, N$, where $\beta_j = \alpha_j(\mu - Z_j)$, $X_i = A_i/a_{N-1}$, and $Y_i = a_i/a_{N-1}$. Similarly, using the relation $P_{11}\alpha_j + P_{12} = 0$ and defining $Z_{N+j} = \overline{Z}_j$ and $\beta_{N+j} = \overline{(\mu + Z_j)/\alpha_j}$, we get the following equation:

$$X_0 + X_1 Z_{N+j} + X_2 Z_{N+j}^2 + \cdots + X_{N-1} Z_{N+j}^{N-1} + X_N Z_{N+j}^N + Y_0 \beta_{N+j} + Y_1 \beta_{N+j} Z_{N+j} + \cdots + Y_{N-2} \beta_{N+j} Z_{N+j}^{N-2} + B_{N+j} Z_{N+j}^{N-1} = 0 \tag{26}$$

These and similar equations for X_i, Y_i can then be written in a compact matrix form

$$MV = W \tag{27}$$

where V, W, and M are as follows:

$$M = \begin{matrix} 1 & Z_1 & Z_1^2 \cdots Z_1^{N-1} & Z_1^N & \beta_1 & \beta_1 Z_1 \cdots \beta_1 Z_1^{N-2} \\ 1 & Z_2 & Z_2^2 \cdots Z_2^{N-1} & Z_2^N & \beta_2 & \beta_2 Z_2 \cdots \beta_2 Z_2^{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & Z_N & Z_N^2 \cdots Z_N^{N-1} & Z_N^N & \beta_N & \beta_N Z_N \cdots \beta_N Z_N^{N-2} \\ 1 & Z_{N+1} & Z_{N+1}^2 \cdots Z_{N+1}^{N-1} & Z_{N+1}^N & \beta_{N+1} & \beta_{N+1} Z_{N+1} \cdots \beta_{N+1} Z_{N+1}^{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & Z_{2N} & Z_{2N}^2 \cdots Z_{2N}^{N-1} & Z_{2N}^N & \beta_{2N} & \beta_{2N} Z_{2N}^2 \cdots \beta_{2N} Z_{2N}^{N-2} \end{matrix}$$

$$V = [X_0, X_1, \ldots, X_N, Y_0, \ldots, Y_{N+2}]^t \tag{28}$$

$$W = [-\beta_1 Z_1^{N-1}, -\beta_2 Z_2^{N-1}, \ldots, -\beta_{N+1} Z_{N+1}^{N-1}, \ldots, -\beta_{2N} Z_{2N}^{N-1}]^t$$

which can be solved at once for X_i and Y_i; for example,

$$T = X_N = \frac{\text{Det } M[W]}{\text{Det } M} \tag{29}$$

where Det $M[W]$ is the $2N \times 2N$ determinant in which a particular column is substituted by the vector W.

4. GAUGE TRANSFORMATION

For case (a) we observe from equation (6) that the u and V matrices of the Lax pair possess first- and second-order poles at $\lambda = \infty$. We demand that in general our Backlund transformation and the associated gauge transformation keep this analytical structure intact. Of course in general the demand is that the analyticity property remains unchanged. From general considerations we can write that $u(\lambda)$, $V(\lambda)$ will have the following structures:

Case (a):

$$u(\lambda) = \lambda\begin{pmatrix} i\alpha & ik \\ -ik & -i\alpha \end{pmatrix} + \begin{pmatrix} ik & ib \\ \pm i\bar{b} & -ik \end{pmatrix} \tag{30}$$

$$V(\lambda) = -2\lambda^2\begin{pmatrix} i\alpha & ik \\ -ik & -i\alpha \end{pmatrix} - 2\lambda\begin{pmatrix} ik & ib \\ \pm ib & -ik \end{pmatrix} + \begin{pmatrix} \pm i|b|^2 & -bx \\ \pm\bar{b}x & \mp i|b|^2 \end{pmatrix} \tag{31}$$

Case (b):

$$u(\lambda) = i\lambda\begin{pmatrix} \alpha & k \\ k & -\alpha \end{pmatrix} + i\begin{pmatrix} k & b_0 \\ \pm b_0 & k \end{pmatrix} \tag{32}$$

$$V(\lambda) = -2i\lambda^2\begin{pmatrix} \alpha & k \\ k & -\alpha \end{pmatrix} - 2i\lambda\begin{pmatrix} k & b_0 \\ \pm\bar{b}_0 & k \end{pmatrix} + \begin{pmatrix} C_0 & -b_{0X} - b_0/x \\ \pm\bar{b}_{0x} \pm \bar{b}_0/x & C_0 \end{pmatrix} \tag{33}$$

where

$$C_0 = i|b_0|^2 \pm 2i \int \frac{|b_0|^2}{x}\, dX$$

If the gauge transformation is effected by the matrix S given in condition (vi), then we get

$$\alpha' = \alpha, \qquad k' = ke^{2i\chi}, \qquad K' = K + \chi_x$$

along with

$$b' = be^{2i\chi}, \qquad \bar{b}' = \bar{b}e^{-2i\chi} \tag{34}$$

So if initially $K = 0$, we must have $K' = 0$, whence $\chi_x = 0$. Similarly we can show that $\chi_t = 0$. So χ must be independent of (x, t). Now we utilize the following condition satisfied by P:

$$u(Z)P(Z) = P_x(Z) + P(Z)u^0(Z) \tag{35}$$

or

$$[ZD + B]\sum_{j=1}^{N} Q_j Z^j = \sum_{j=1}^{N} Q_{jx} Z^j + \sum_{j=1}^{N} Q_j Z^j[ZD^0 + B^0]$$

where

$$B = \begin{pmatrix} i\alpha x/(4t) + ik & ik + ib \\ -ik \pm i\bar{b} & -i\alpha x/(4t) - ik \end{pmatrix}$$

Here the function for which the solution is to be found is $q = b' = be^{2i\chi}$.
We have

$$B^0 = \begin{pmatrix} ix/4t & iq_0 \\ \pm i\bar{q}_0 & -ix/4t \end{pmatrix}, \qquad D = \begin{pmatrix} i\alpha/t & ik/t \\ -ik/t & -i\alpha/t \end{pmatrix} \tag{36}$$

$$D^0 = \begin{pmatrix} i/t & 0 \\ 0 & -i/t \end{pmatrix}, \qquad Q_N = \begin{pmatrix} \bar{A}_N & \bar{a}_{N-1} \\ -a_{N-1} & A_N \end{pmatrix}$$

From equation (35) and the properties of b, k, α, and χ we get

$$q = \frac{i\bar{A}_{Nx}e^{2i\chi} \mp \bar{q}_0\bar{a}_{N-1}e^{2i\chi}}{a_{N-1}} \tag{37}$$

where we have taken $\alpha = \alpha' = 1$ and $k = k' = 0$, and $q = b' = be^{2i\chi}$. Now let us recall equation (29) and observe that with the help of the expression for $\bar{T}$ we can rewrite (37) as

$$q = \frac{i(\bar{a}_{N-1}\bar{T}_x + \bar{T}\bar{a}_{N-1,x}) \mp \bar{q}_0\bar{a}_{N-1}}{a_{N-1}} e^{2i\chi} \tag{38}$$

Also the condition that Det $Q_N = 1$ leads to

$$\bar{a}_{N-1}a_{N-1} = (1 + T\bar{T})^{-1} \tag{39}$$

5. PARTICULAR SOLUTIONS

To prove the viability of our formulas (38), we show here that if we start from the trivial solution $q_0 = 0$, then in both cases (a) and (b) we get the one-soliton solution which was obtained by other methods previously. If we set $q_0 = 0$, then from equation (38) we get

$$q = \frac{i[\bar{T}(\bar{T}T + 1)^{-1/2}]_x}{(\bar{T}T + 1)^{-1/2}} e^{2i\chi(xt)} \tag{40}$$

For $N = 1$ we get

$$\bar{T} = \frac{(\mu + Z_1)/\alpha_1 - (\bar{\mu} - \bar{Z}_1)\bar{\alpha}_1}{\bar{Z}_1 - Z_1} \tag{41}$$

where the space-time variation of α_1 is given by, for case (a),

$$\alpha_{1x} = 2i\left(\frac{Z_1}{t} + \frac{x}{4t}\right)\alpha_1 \tag{42}$$

$$\alpha_{1t} = -4i\left(\frac{Z_1^2}{t^2} + \frac{x}{16t^2} + \frac{Z_1 x}{2t^2}\right)\alpha_1$$

It is not difficult to observe a common solution of (42) and given by

$$\alpha_1 = \exp\left[\frac{i}{t}\left(2Z_1 + \frac{x}{2}\right)^2\right] \tag{43}$$

Now we take $Z_1 = a + ib$, and using this expression for α_1 in (43), we get

$$q(xt) = \frac{e^{2i\chi}}{2t}\exp\left\{-\frac{i}{t}\left[\frac{x^2}{4} + 2ax + 4(a^2 - b^2)\right]\right\}$$

$$\times\left\{N\exp\left[\frac{2b}{t}(4a + x)\right] + M\exp\left[-\frac{2b}{t}(4a + x)\right]\right\} \tag{44}$$

where

$$N = \left[\frac{i(a + \mu)}{b} - 1\right]\left(\frac{4a + x}{2} + 2ib\right)$$

$$M = \left[\frac{i(a - \bar{\mu})}{b} + 1\right]\left(\frac{4ax}{2} - 2ib\right)$$

which is the form of the one-soliton solution obtained in De and Chowdhury (1991) by the Riemann–Hilbert transform.

Let us now consider case (b). In this situation we again choose $q_0 = 0$, so that

$$\alpha_{jx} = \frac{ix}{2(Z_j + t)}\alpha_j \tag{45}$$

$$\alpha_{jt} = -\frac{ix^2}{4(Z_j + t)^2}\alpha_j$$

Again a common solution can be obtained and we get

$$\alpha_j = K\exp\left[\frac{ix^2}{4(Z_j + t)}\right]$$

Now proceeding as before, we obtain

$$q(xt) = \frac{A_{0x} + A_{1x}Z_1}{(\mu - Z_1)a_0}e^{2i\chi} \tag{46}$$

where

$$\frac{A_1}{a_0} = \frac{(\mu + Z_1)(\alpha_1)^{-1} - (\bar{\mu} - \bar{Z}_1)\bar{\alpha}_1}{(\bar{Z}_1 - Z_1)} \tag{47}$$

Again assuming $Z_1 = a + ib$, after considerable amount of algebra we get

$$q(xt) = (\theta_1 + i\theta_2)(\xi_1 + i\xi_2)e^{\Lambda} \tag{48}$$

where

$$\begin{aligned}
\Lambda &= -\frac{bx^2}{m} + i\left[2\chi - \frac{x^2(a+t)^2}{m}\right] \\
\theta_1 &= \frac{(\mu_1^2 - a^2) + (\mu_2^2 - b^2)}{(\mu_1 - a)^2 + (\mu_2 - b)^2} \\
\theta_2 &= \frac{2b\mu_1 - 2a\mu_2}{(\mu_1 - a)^2 + (\mu_2 - b)^2} \\
\xi_1 &= \frac{2bx}{m}\,\frac{C}{C + Ae^{-2bX^2/m} + Be^{2bX^2/m}} \\
\xi_2 &= \frac{2x(a+t)}{m}
\end{aligned} \tag{49}$$

6. DISCUSSION

In the above analysis we have shown how the N-fold Backlund transformation can be constructed for the deformed NLS equations in general. The procedure has been illustrated with two very important cases. For $N = 1$ and zero seed solution we at once obtain the one-soliton solution. It may be observed that the solution obtained in case (a) is valid everywhere except at $t = 0$, where it has a singularity, which was also present in the equation itself. In the second case (b), the situation is slightly different. For example, this solution can become infinite for some values of t, depending upon the choice of the parameter values.

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